# Critical dimensionalities of phase transitions on fractals

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Several arguments are given leading to the sufficient and necessary condition for spontaneous symmetry breaking at a finite temperature on fractals, which is $\tilde{d} \geq 2$ for discrete symmetry and $d \geq d_w + 1$ for continuous symmetry, where $\tilde{d}$, $d$, and $d_w$ are, respectively, the spectral dimensionality, fractal dimensionality, and dimensionality of the random walk of this structure. In addition, phase transitions can always occur at $T_c > 0$ on infinitely ramified lattices. Since $\tilde{d} < 2$ for fractals usually studied, $T_c$ was always found to be 0 on finitely ramified fractals. $\tilde{d} \geq 2$ can be satisfied by a bifractal, a Cartesian product of two fractals, hence $T_c > 0$ is expected. A Peierls-Griffiths proof is given for an Ising model on an example of bifractals, the periodic Koch lattice with $\tilde{d} = 2$, showing that $T_c$ is indeed finite. A unified picture concerning both fractal and Euclidean lattices is thus obtained.

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## I. INTRODUCTION

It is well known that the dimensionality of space $D$ plays an important role in phase transitions. The critical $D$ above which a continuous symmetry is spontaneously broken at a finite temperature is 2 [1–3]. For a discrete symmetry such as that of an Ising model a nonzero critical temperature ($T_c$) is expected also in two dimensions [4,5,3]. These results concern translationally invariant systems. Then, how about non-translationally-invariant systems, such as fractals [6]?

Naively one expects for phase transitions on fractals that the above role of $D$ is replaced by the fractal dimensionality $d$. However, things are not so. It was found by Gefen and co-workers that Ising models on infinitely ramified fractals undergo a phase transition at $T_c > 0$ even if its fractal dimensionality is less than 2, while $T_c$ is always found to be zero on finitely ramified fractals regardless of their fractal dimensionalities [7]. Here the concept of ramification comes into play. According to Ref. [6], the order of ramification ($R$) involves the cut set containing the *smallest* number of points that must be removed in order to disconnect the set $S$; it involves separately the neighborhood of every point in $S$. If $S$ is a Sierpinski gasket, $R$ can be either $3 = R_{\min}$ or $4 = R_{\max}$. For a standard Euclidean lattice, $R$ attained on lattice sites is 4 (square), 6 (triangle), or 3 (hexagon). On the other hand, a Sierpinski carpet is infinitely ramified. It was presented as a general rule that Ising systems have $T_c = 0$ on finitely ramified lattices, and $T_c > 0$ on infinitely ramified lattices [7]. We think that this is unsatisfactory, since on the Euclidean lattices $T_c$ can be nonzero. Besides, one expects a unified rule applying on both fractal and Euclidean lattices.

As a matter of fact, phase transitions are governed by long-range correlations, i.e., low-frequency modes. Hence, as pointed out by Dhar [8], the spectral dimensionality $\tilde{d}$ [8,9] should be more relevant to phase transitions than the fractal dimensionality. Recently it was proved by Cassi that classical O($n$) and quantum Heisenberg ferromagnetic models cannot have spontaneous magnetization at any finite temperature if random walks on the same structure are recursive, i.e., $\tilde{d} \leq 2$ [9].

In this paper we discuss under what conditions symmetries, including discrete and continuous ones, can be spontaneously broken on fractals. For continuous symmetry, a Peierls-Landau-type approach is given in Sec. II leading to the same result as above, while the Peierls-type argument in Sec. III leads to a criterion consistent with but a bit stricter than this. Also in Sec. III, it is shown for the Ising model that the phase transition occurs at a finite temperature iff $\tilde{d} \geq 2$, which reduces to $D \geq 2$ on Euclidean lattices. The spectral dimensionalities of fractal lattices usually studied are smaller than 2. It is pointed out in Sec. V that $\tilde{d} \geq 2$ can be satisfied by a class of self-affine fractal lattices, the so-called bifractal lattices. The Peierls-type arguments are extended on a bifractal. In Sec. V we generalize the Peierls-Griffiths proof [4,5,3] to Ising model on a bifractal lattice with $\tilde{d} = 2$, proving rigorously that $T_c$ is indeed nonzero.

## II. PEIERLS-LANDAU-TYPE APPROACH

One way to find out whether an ordered phase can exist is the so-called Peierls-Landau-type approach, that is, to see if it is stable against long-wavelength fluctuations [3]. Consider a lattice and let $u(x)$ be a displacement vector, which gives the deviation from equilibrium. $u(x)$ can be decomposed into normal modes, the number of which is determined by $\tilde{d}$ [9]. The energy residing in a normal mode is given by

$$E_k = \tfrac{1}{2}\omega^2(k)|q(k)|^2 , \tag{1}$$

where $q(k)$ is the Fourier transform of $u(x)$. By the

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equipartition of energy, the average value of $E_k$ is estimated to be $k_B T$. Hence the mean-square amplitude of a normal mode is

$$\langle |q(k)|^2 \rangle = \frac{2k_B T}{\omega^2(k)} . \quad (2)$$

Therefore the mean-square excitation is

$$\langle u^2 \rangle = \int (dk) \frac{2k_B T}{\omega^2(k)} . \quad (3)$$

Phase transitions are dominated by low-frequency modes (Goldstone excitations). In this region, it is expected for elastic vibration that $\omega \sim k$. Thus the integral (3) behaves near the lower limit like $\int dk\, k^{\tilde{d}-3}$, which converges only if $\tilde{d} > 2$. This means that the spatial order will be destroyed by long-wavelength fluctuations in lattices with $\tilde{d} \leq 2$. Similarly, spontaneous magnetization cannot happen in a continuous media with $\tilde{d} \leq 2$, because the magnetic order would have been destroyed by spin waves. This result is just what Cassi proved [10]. This condition is expected to be relaxed for a discrete symmetry, as shown below.

## III. PEIERLS-TYPE ARGUMENT

Another approach to this problem is the so-called Peierls-type argument, that is, to examine whether long-range order is stable against the formation of domain walls, the boundaries between regions with different values of order parameter [4,5,3]. Here we present a generalization of it applicable to both fractal and Euclidean lattices.

First we consider Ising model on a lattice whose fractal dimensionality is $d$. At zero temperature, the interactions tend to order the spins, thus the ground state is in a definite ferromagnetic pure phase. We now examine whether this state resists the formation at a low but finite temperature of a domain of opposite spins, which will increase the energy but has some entropy, too; thus the sign of the free energy is not obvious.

If the linear size of the domain is $L$, the increase of energy, which is only due to the spins that interact across the domain surface, is proportional to $L^{d-1}$. The domain surface can be made by a random walk on the lattice. One needs $\sim L^{d_w}$ steps to make a volume of linear size $L$, hence $L^{d_w-1}$ steps are needed to make the surface. Consequently the number of ways in which the domain is created is proportional to $c^{L^{d_w-1}}$, where $c$ is the chances of each step of the random walk. The entropy is thus proportional to $L^{d_w-1}$. For the free energy to be positive at a low temperature, we should have $d \geq d_w$, or, say, $\tilde{d} = 2d/d_w \geq 2$, including the marginal case $\tilde{d} = 2$. Hence this condition is more relaxed than for continuous symmetry.

For a system with continuous-order parameter $m(x)$, the domain wall is of finite thickness, in which $m(x)$ changes continuously from the value inside to outside. The energy cost of the domain wall is estimated to be

$$U = \int_{\text{wall}} (dx) |\nabla m(x)|^2 \propto \frac{(\text{volume of wall})}{L^2} \propto L^{d-2} , \quad (4)$$

where the estimate is used that $|\nabla m(x)|^2 \propto L^{-2}$ and the thickness of the wall is proportional to $L$, since $L$ is the only relevant length. Therefore we may find that, if $\tilde{d} \leq 2$, then the equality $d-2 < d_w - 1$ is satisfied; hence the entropy favors domain creation, and thus there is no phase transition at finite temperature. This is just what Cassi proved [10] and has been shown in the preceding section. It is interesting to note that $\tilde{d} \leq 2$ is only the sufficient but not necessary condition for $T_c = 0$. Comparing the energy and entropy, we may find that one needs only $d < d_w + 1$ to obtain $T_c = 0$. Thus for continuous symmetry to be spontaneously broken at a finite temperature, the condition is $d \geq d_w + 1$, which is stricter than $\tilde{d} > 2$. We should point out that this argument is not rigorous and it is interesting to test and prove this.

If the order of ramification is infinite, the cost of energy is infinite to create a domain wall. Thus the free energy cannot be negative and $T_c$ could be nonzero regardless of the dimensionalities.

## IV. BIFRACTALS

For ordinary fractals $\tilde{d} < 2$. This is why $T_c$ is always found to be zero on finitely ramified fractals. It seems difficult to construct fractal lattices with $\tilde{d} \geq 2$. Recently we found that it could be done by making a Cartesian product of two fractals, and we refer to such a structure as a "bifractal" [11]. The Cartesian product of two graphs is defined as follows. Let $V(A)$ and $V(B)$ represent the vector sets of graphs $A$ and $B$. If graph $C$ is the Cartesian product of $A$ and $B$, then its vertex set $V(C)$ consists of all pairs $(i,j)$ where $i \in V(A)$ and $j \in V(B)$. Adjacency on $C$ must also be defined. Let $(i,j)$ and $(k,l)$ be adjacent on $C$ if either $i$ is adjacent to $k$ on $A$ and $j = l$, or $i = k$ and $j$ is adjacent to $l$ on $B$. Hence, for example, the square lattice is a Cartesian product of two linear chains. On a bifractal, there exists a global spectral dimensionality, which is the sum of those of subfractals and thus can be $\geq 2$ [11]. Bifractals represent a class of self-affine fractals, which have nontrivial global fractality unlike those characterized by self-affine functions. For instance, a variety of directed growth models lead to self-affine aggregates, which could be viewed as the products of self-similar fractals. The typical example is the directed percolation cluster at criticality, with different fractal dimensions along and perpendicular to the composed direction [12]. In Fig. 1 we construct an example of bifractal lattices, the "periodic Koch lattice" (PKL), which is formed by periodically arrayed Koch curves (on $xy$ planes) along the direction perpendicular to them ($z$ direction). Its spectral dimensionality is 2, since that of the linear chain and the Koch curve are both 1.

Let $R_1$ and $R_2$ be the linear sizes that a random walker reaches on the two subfractals, respectively, after $N$ steps. Then $N \propto R_1^{d_w^{(1)}} \propto R_2^{d_w^{(2)}}$. Thus the volume it reaches is $\sim R_1^{d_1} R_2^{d_2} \propto N^{d_1/d_w^{(1)}} N^{d_2/d_w^{(2)}} = N^{(\tilde{d}_1 + \tilde{d}_2/2)} = N^{\tilde{d}/2}$, where $d_i$, $d_w^{(i)}$, and $\tilde{d}_i$ ($i=1,2$) are corresponding quantities of subfractals. Hence the condition that a random walk is transient is still $\tilde{d} > 2$. However, some modification are needed to apply the above Peierls-type argument on a bi-

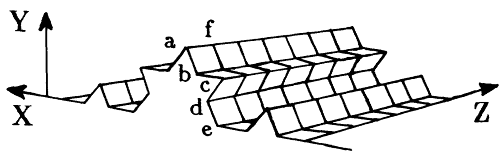


FIG. 1. Periodic Koch lattice, with a Cartesian coordinate system on it.

fractal, where global scaling of random walks is generally absent.

Now we consider Ising model on a bifractal. Assume the domain is created by an $N$-step random walk, the linear sizes reached are $L_1$ and $L_2$, respectively, on the two subfractals, with $L_1^{d_w^{(1)}} \propto L_2^{d_w^{(2)}} \propto N$. Generally, for a volume $V$ with sizes $L_1$ and $L_2$ on the two subspaces, respectively, the area of the surface is proportional to $(V/L_1+V/L_2)$. Thus it can be estimated that the entropy is proportional to $L_1^{d_w^{(1)}-1}+L_2^{d_w^{(2)}-1}$ and the energy cost is proportional to $L_1^{d_1+(d_2d_w^{(1)}/d_w^{(2)})-1}+L_2^{d_2+(d_1d_w^{(2)}/d_w^{(1)})-1}$. One may find that it still holds that there is a phase transition in the Ising model at a finite temperature iff $\tilde{d} \geq 2$.

Then we turn to the model with a continuous-order parameter $m(x)$. In the present situation, there are two length scales on the two independent subspaces, therefore $|\nabla m|^2 \sim 1/L_1^2+1/L_2^2$. Consequently the energy cost of creating the domain wall is thus $L_1^{d_1+(d_2d_w^{(1)}/d_w^{(2)})-2}+L_2^{d_2+(d_1d_w^{(2)}/d_w^{(1)})-2}$. It can be found that the entropy dominates iff

$$d_1+\frac{d_2d_w^{(1)}}{d_w^{(2)}}<d_w^{(1)}+1 ,$$

and (5)

$$d_2+\frac{d_1d_w^{(2)}}{d_w^{(1)}}<d_w^{(2)}+1 ,$$

which reduces to $d<d_w+1$, where $d=d_1+d_2$, when $d_w^{(1)}=d_w^{(2)}=d_w$. The above inequalities are obviously valid when $\tilde{d} \leq 2$. Still, as in the preceding section, a stricter constraint is given for $T_c$ to be finite in models with continuous symmetry. The qualitative physical picture remains unchanged on a bifractal from that on an isotropic fractal.

## V. PEIERLS-GRIFFITHS PROOF ON A BIFRACTAL WITH $\tilde{d}=2$

In this section, we focus on the Ising model on the periodic Koch lattice (Fig. 1). Since its spectral dimensionality is 2, according to Sec. III, there is a spontaneous magnetization below a finite temperature. Now we prove this rigorously by generalizing the Peierls-Griffiths proof about the two-dimensional Ising model [4,5,3].

The Ising model on a Koch curve, and consequently on a PKL, can be defined in two ways. In the first way, the interactions are restricted to nearest neighbors along the curve, that is, to segments $\langle ab \rangle$, $\langle bc \rangle$, $\langle cd \rangle$, and $\langle de \rangle$, but not along $\langle bd \rangle$. In the second way, which is more realistic, the nearest neighbors are defined on the whole space, i.e., there is also an interaction on $\langle bd \rangle$. We assume that the nearest-neighbor interactions are isotropic, i.e., that in the $z$ direction (along $\langle af \rangle$) is the same as that in the $xy$ plane (along $\langle ab \rangle$). It is just the wiggled two-dimensional Ising model on a square lattice in the first case, and thus the Peierls-Griffith proof [4,5,3] applies directly if we use a non-Cartesian coordinate system, one dimension of which is along the $z$ direction, another of which is along the Koch curves. Hence a spontaneous magnetization is expected at a finite temperature. In the following, a proof is given in the Cartesian coordinate system (Fig. 1) in order to apply also in the second case. Some parts of it are almost the same as Refs. [5] or [3], but have to be reproduced for completeness.

Consider an arbitrary configuration on the lattice. Similar to what was done on a two-dimensional square lattice [5,3], a domain wall is a continuous line drawn between up (+) spins and down (−) spins, since the topological dimensionality of a PKL is also 2. The length of wall can be defined unambiguously as follows. It is the relative $z$ coordinates, and it is just the length of the line, when the line is along the $z$ axis, while it is the relative $x$ coordinates when the line is on a Koch curve. Any point on the PKL can be characterized by $x$ and $z$ coordinates. The domain walls are drawn in the conventional sense, i.e., to let − spins always lie to the right of the wall and + spins to the left. Where there is an ambiguity, it bends to the right.

The probability $P\{s\}$ of occurrence of the configuration $\{s\}$ is

$$P\{s\}=\frac{e^{-\beta E\{s\}}}{\sum_{\{s\}} e^{-\beta E\{s\}}} , \tag{6}$$

where the sum extends over all configurations. For any configuration, the average magnetization per spin is defined as

$$M=\frac{M_+-N_-}{2N} , \tag{7}$$

where $N_+$ and $N_-$ are the number of + and − spins in the configuration. Obviously the average of $M$ taken over all configurations weighted with probability (6) is zero due to symmetry. Following Griffiths [5], we define the spontaneous magnetization by

$$M_0=\lim_{N\to\infty} \langle |M| \rangle , \tag{8}$$

where the angular brackets denote the thermal average. Also, we shall prove that, at sufficiently low temperature,

$$\langle |M| \rangle \geq M_1 > 0 , \tag{9}$$

independent of $N$.

As the first step, we impose the boundary condition that all spins on the boundary are +, and obtain an upper bound to $N_-$, and thus a lower bound to $\langle M \rangle$, in this case. Now every domain wall is a closed curve. Consider the set of all closed domain walls. They are classified according to length $b$, and each is given a number $i$ within a class of given length. Thus any domain wall is uniquely characterized by a label $(b,i)$. There are various distributions between the lengths in $z$ and $x$ directions for a certain $b$, constrained by

$$2(x+z)=b \ . \tag{10}$$

The number of $-$ spins inside a closed curve is just the volume enclosed,

$$V=x^D z \ , \tag{11}$$

where $D$ is the fractal dimensionality of the Koch curves. From (10), we may find that

$$V \le \left(\frac{b}{2}\right)^{D+1} \frac{D^D}{(D+1)^{D+1}} \ . \tag{12}$$

The number of domain walls of length $b$ is bounded by [5]

$$m(b) \le \frac{4N3^b}{3b} \ . \tag{13}$$

Every $-$ spin is enclosed by at least one domain wall, since the spins on the boundary are all $+$. In a particular configuration, let $X(b,i)$ be 1 if the domain wall $(b,i)$ occurs in that configuration, and 0 otherwise. Then in this configuration the number of $-$ spins satisfies

$$N_- \le \sum_b \left(\frac{b}{2}\right)^{D+1} \frac{D^D}{(D+1)^{(D+1)}} \sum_{i=1}^{m(b)} X(b,i) \ . \tag{14}$$

The thermal average of $X(b,i)$ is

$$\langle X(b,i)\rangle = \sum_{\{s\}}{}' P\{s\} \ , \tag{15}$$

where the prime indicates that the sum is restricted to those configurations in which $(b,i)$ occurs. If $C$ is a configuration in which $(b,i)$ occurs, let $C^*$ be the configuration obtained from $C$ by reversing every spin inside the domain wall. Their energies are related by

$$E_C = E_{C^*} + 4\varepsilon(x^D + z) \tag{16}$$

for the Ising model defined in the first way, i.e., without interaction on $\langle bd \rangle$. $\varepsilon$ is the nearest-neighbor ferromagnetic interaction. In the Ising model defined in the second way, there are perhaps two nearest-neighbor interactions, e.g., those along $\langle bc \rangle$ and also $\langle bd \rangle$, across the wall in the $z$ direction. In this case, one should have

$$E_{C^*} + 4\varepsilon(x^D + z) \le E_C \le E_{C^*} + 4\varepsilon(x^D + 2z) \ . \tag{17}$$

Since $D \ge 1$, one obtains in any case

$$E_C \ge E_{C^*} + 2\varepsilon b \ . \tag{18}$$

Hence we may find an upper bound on $\langle X(b,i)\rangle$,

$$\langle X(b,i)\rangle \le e^{-2\beta\varepsilon b} \ . \tag{19}$$

Taking the thermal average of (14), we obtain

$$\begin{aligned} \frac{\langle N_- \rangle}{N} &\le \sum_b 4\left(\frac{b}{2}\right)^{D+1} \frac{D^D}{(D+1)^{D+1}} \frac{3^{b-1}}{b} e^{-2\beta\varepsilon b} \\ &< \frac{1}{6}\sum_b b^2 3^b e^{-2\beta\varepsilon b} \\ &= \frac{8}{3}\frac{\kappa^2}{(1-\kappa)^3}\left(1-\frac{3\kappa}{4}+\frac{\kappa^2}{4}\right) , \end{aligned} \tag{20}$$

provided $\kappa = 9e^{-2\beta\varepsilon b}$. We have used the fact that $D<2$. The above ratio is independent of $N$ and, for example, is less than $\frac{1}{2}$ for sufficiently large but finite $\beta$.

Now let us turn to estimate $\langle |M| \rangle$ without imposing constraints upon boundary spins. Each domain wall divides all the spins on the lattice into two sets, that lying to the right and that lying to the left. If it is closed, one set of spins lies inside it and one set lies outside. If it is not closed, we define the smaller of the two sets as lying "inside" and the larger as lying outside. If the two sets contain equal numbers of spins, the set to the right of the wall will be said to be "inside." There are at most $D^D[b/(D+1)]^{D+1}$ spins lying inside a domain wall of length $b$. An upper bound to the probability of occurrence of the $i$th domain wall is again given by (13). The configurations may be divided into two classes, in those belonging to one of which, denoted by $A$, all spins lie inside some domain walls. In configurations belonging to the other class, denoted by $B$, there is at least one $-$ spin which lies outside all domain walls, hence all $+$ spins lie inside at least one domain wall. Therefore

$$\begin{aligned} \langle |M| \rangle &= \sum_{\{s\}} |M\{s\}| P\{s\} \\ &\ge \sum_{\{s\}} [A] M\{s\} P\{s\} - \sum_{\{s\}} [B] M\{s\} P\{s\} \\ &= \tfrac{1}{2} - \frac{1}{N}\left(\sum_{\{s\}} [A] N_-\{s\} P\{s\} \right. \\ &\qquad \left. + \sum_{\{s\}} [B] N_+\{s\} P\{s\} \right) , \end{aligned} \tag{21}$$

where $M\{s\}$, $N_-\{s\}$, and $N_+\{s\}$ are the quantities in the configuration $\{s\}$ and $\sum_{\{s\}}[A]$ and $\sum_{\{s\}}[B]$ denote sums over configurations of class $A$ or $B$. For configurations in class $A$,

$$N_- \le \sum_b [A] b^{D+1} \frac{D^D}{(D+1)^{D+1}} \sum_{i=1}^{m(b)} X(b,i) \ . \tag{22}$$

The same inequality holds for $N_+$ for all configurations in class $B$. Therefore

$$\begin{aligned} &\sum_A N_-\{s\} P\{s\} + \sum_B N_+\{s\} P\{s\} \\ &\quad \le \sum_b b^{D+1} \frac{D^D}{(D+1)^{D+1}} \sum_{i=1}^{m(b)} X(b,i) \\ &\quad \le \frac{4}{3}\sum_b b^2 3^b e^{-2\beta\varepsilon b} \\ &\quad = \frac{64}{3}\frac{\kappa^2}{(1-\kappa)^3}\left(1-\frac{3\kappa}{4}+\frac{\kappa^2}{4}\right) , \end{aligned} \tag{23}$$

with $\kappa = 9e^{-2\beta\varepsilon}$. Thus, if the temperature is low enough, we obtain a lower bound of the form (9), independent of $N$.

Such is our proof. If the interactions are anisotropic, i.e., that along $\langle af \rangle$ is different from that along $\langle ab \rangle$, the inequality (18) is still valid given ε as the smaller one. hence the above proof still applies. Also, the proof may be generalized to other bifractals whose subfractal on the $xy$ plane is one of the other quasilinear lattices. In an earlier discussion on Ising model on the PKL using a bond-moving approximation [13], an indication was found that $T_c$ is nonzero.

## VI. SUMMARY

In this paper, we address under what conditions phase transitions occur at finite temperature on general graphs, such as fractals. This is determined by the geometric (fractal dimensionality $d$) together with the diffusion (dimensionality of random walks $d_w$), and thus the dynamic (spectral dimensionality $\tilde{d}$) properties of the lattice. In a Euclidean lattice, the dimensionality of the random walk is trivially 2, while the dimensionality of space is identical with the fractal and also the spectral dimensionality. Hence one might misunderstand that the value of dimensionality of space determines if phase transitions occur at a finite temperature. Studies on phase transitions on fractal lattices clarify this matter and confirm that it is long-wavelength modes that dominate phase transitions. Generalizing some famous arguments, we have shown that, on finitely ramified lattices, the sufficient and necessary condition for phase transitions to occur at finite temperature is, respectively, that $\tilde{d} \geq 2$ and $d \geq d_w + 1$ in models with discreet and continuous symmetries. $T_c$ is always nonzero on infinitely ramified systems. The spectral dimensionalities of fractal lattices usually studied are less than 2; hence $T_c$ was found to be zero on finitely ramified fractals. We point out that $\tilde{d} \geq 2$ can be satisfied on a class of self-affine fractals, the bifractals, which are Cartesian products of two fractals. Because of the absence of global scaling for random walks on a bifractal, the condition $d \geq d_w + 1$ should be generalized to a more general one where the dimensionalities of two subfractals are involved. As a test of the results presented here, we will study in detail phase transitions on bifractals. It is also interesting to study the field-theoretic correspondence.

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